Ant Travels in Bugniverses

How to solve it in eight easy steps (with a few sub-steps)

1. this is clearly, a *greatest common divisor* problem with a twist: We are asked to find x, y such that
   (a) the equation $ax + by = c$ holds, and
   (b) x must be the least positive possible value *i.e.* $x > 0$ and, if $au + bv = c$ and $u > 0$ then $u \geq x$.
2. this kind of equation either have 0 (zero) or ∞ (infinite) solutions:
   - it should be easy to check that if
     $$a = 4, b = 6, c = 9$$
     there is no solution because the LHS is *even* and the RHS is *odd*.
   - but, if $c = 18$, we have
     $$x = 0, y = 3 \; ; \; x = 3, y = 1 \; ; \; \text{etc.}$$
3. the criterium for solution existence is $d|c$, where $d = \gcd(a, b)$
4. the values of d, x, y can be found using one of the variants of euclid's algorithm
5. in general, if $d|c$, the equation can be solved in two steps:
   (a) use the *euclid's algorithm* to find x', y' such that
   $$ax' + by' = d$$
   (b) since $d|c$ we have $c = \gamma d$ and set
   $$x_0 = x'\gamma \; , \; y_0 = y'\gamma$$
6. although this assures that $ax_0 + by_0 = c$, the condition on x might fail...
7. to find the right value of x we must understant how a change in the "ax" part of the equation affects the "by" part:
   (a) since $d = \gcd(a, b)$ and $d|c$, we have
   $$a = \alpha d \; , \; b = \beta d \; , \; c = \gamma d$$
   (b) the previous equation can be rewritten
   $$\begin{aligned} ax_0 + by_0 &= c \Leftrightarrow \\ \alpha x_0 + \beta y_0 &= \gamma \Leftrightarrow \\ \alpha x_0 + (Z - Z) + \beta y_0 &= \gamma \Leftrightarrow \\ \alpha(x_0 + A) + \beta(y_0 - B) &= \gamma \end{aligned}$$
   (c) so we have
   $$\alpha A = Z = \beta B$$
   (d) since $\gcd(\alpha, \beta) = 1$, it must be
   $$A = \beta k \; , \; B = \alpha k$$
   because $Z = \alpha A = \alpha\beta k = \beta\alpha k = \beta B$
   (e) now we have the rule to change x, y in $ax_0 + by_0 = c$:
   $$\begin{aligned} x_k &= x_0 + \beta k \\ y_k &= y_0 - \alpha k \end{aligned}$$
   (f) so, any solution x_k, y_k is related to x_0, y_0 by
   $$\begin{aligned} x_k &\equiv x_0 \pmod{\beta} \\ y_k &\equiv y_0 \pmod{\alpha} \\ k &= \frac{y_0 - y_k}{\alpha} = \frac{x_k - x_0}{\beta} \end{aligned}$$
8. therfore, we need only to reduce $x_0 (\mathrm{mod} \beta)$ not forgetting two details:
   (a) if $x = x_0 \% \beta \leq 0$ we need to add an extra β, so that it becomes positive;
   (b) to each β added to the ax part corresponds an α subtracted in the by part; and the number of β's "added" is ... (exercise);

Appendix

Euclid's algorithm with additional computation of x,y:

```
def gcd( a, b, &x, &y ):
    if ( b > a ) return gcd( b, a, y, x )
    if ( b == 0 ):
        x, y = 1, 0
        return a
    d = gcd( b, a % b, x1, y1 )
    x, y = y1, x1 - floor( a / b ) * y1
    return d
```

from Skiena, S. and Revilla, M, *Programming Challenges*, Springer